Diferenciranje pod znakom integrala

Neka je dat integral koji zavisi od parametra α:

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, dx$$

Ako su $f(x,\alpha)$, $f'(x,\alpha)$ neprekidne f-je, ako postoje $b'(\alpha)$ i $a'(\alpha)$ tada

$$I'(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f'_\alpha(x,\alpha)\, dx + b'(\alpha)\, f(b(\alpha), \alpha) - a'(\alpha) \cdot f(a(\alpha), \alpha)$$

Ako granice a i b ne zavise od α tada

$$I'(\alpha) = \int_{a}^{b} f'_\alpha(x,\alpha)\, dx$$

(#) Polazeći od integrala $\int_0^b \frac{dx}{1+\alpha x}$ izračunati

$$\int_0^b \frac{x\,dx}{(1+\alpha x)^2} \quad i \quad \int_0^b \frac{x^2\,dx}{(1+\alpha x)^3}.$$

Rj.

$$I(\alpha) = \int_0^b \frac{dx}{1+\alpha x} = \left| \begin{array}{ll} 1+\alpha x = t & x=0 \Rightarrow t=1 \\ \alpha\,dx = dt & x=b \Rightarrow t=1+\alpha b \\ dx = \frac{1}{\alpha}dt & \end{array} \right| =$$

$$= \frac{1}{\alpha}\int_1^{1+\alpha b} \frac{dt}{t} = \frac{1}{\alpha}\ln|t|\Big|_1^{1+\alpha b} = \frac{1}{\alpha}\ln|1+\alpha b|$$

$$f'_\alpha(x,\alpha) = \left(\frac{1}{1+\alpha x}\right)'_\alpha = (-1)(1+\alpha x)^{-2}\cdot x = \frac{-x}{(1+\alpha x)^2}$$

$$I'(\alpha) = \int_a^b f'_\alpha(x,\alpha)\,dx \Rightarrow \quad I'(\alpha) = \int_0^b \frac{-x}{(1+\alpha x)^2}\,dx \Rightarrow$$

$$\Rightarrow \int_0^b \frac{x\,dx}{(1+\alpha x)^2} = -I'(\alpha)$$

Kako je $I(\alpha) = \frac{1}{\alpha}\ln|1+\alpha b|$ to je $I'_\alpha = -\frac{1}{\alpha^2}\ln|1+\alpha b| + \frac{1}{\alpha}\cdot\frac{1}{1+\alpha b}\cdot b$

Prema tome

$$\int_0^b \frac{x\,dx}{(1+\alpha x)^2} = \frac{1}{\alpha^2}\ln|1+\alpha b| - \frac{b}{\alpha(1+\alpha b)}$$

Slično bi imali $I''(\alpha) = \int_0^b \left(\frac{-x}{(1+\alpha x)^2}\right)'_\alpha dx = \int_0^b \frac{2x^2}{(1+\alpha x)^3}\,dx \Rightarrow$

$$\Rightarrow \int_0^b \frac{x^2}{(1+\alpha x)^3}\,dx = \frac{1}{2}I''(\alpha), \quad I''_\alpha = (I'_\alpha)' = \frac{2}{\alpha^3}\ln|1+\alpha b| + \left(-\frac{1}{\alpha^2}\right)\frac{b}{1+\alpha b}$$

$$-\frac{b}{\alpha^2(1+\alpha b)} + \frac{1}{\alpha}\cdot\frac{-b}{(1+\alpha b)^2}\cdot b \Rightarrow \int_0^b \frac{x^2}{(1+\alpha x)^3}\,dx = \frac{1}{\alpha^3}\ln|1+\alpha b| - \frac{b}{\alpha^2(1+\alpha b)}$$

$$-\frac{b^2}{\alpha(1+\alpha b)^2}$$ traženo rješenje

(#) Izračunati pomoću diferenciranja po parametru integral

$$I(\alpha)=\int_0^{\frac{\pi}{2}}\ln(\sin^2x+\alpha^2\cos^2x)\,dx,\quad \alpha>0.$$

Rj. Ako je data f-ja dvije promjenjive $f(x,\alpha)$, ako su $f(x,\alpha)$ i $f'_\alpha(x,\alpha)$ neprekidne f-je tada za integral $I(\alpha)=\int_a^b f(x,\alpha)\,dx$ vrijedi $I'_\alpha(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx$.

f'_α – predstavlja izvod f-je f po promjenjivoj α

$$I(\alpha)=\int_0^{\frac{\pi}{2}}\ln(\sin^2x+\alpha^2\cos^2x)\,dx$$

$$f(x,\alpha)=\ln(\sin^2x+\alpha^2\cos^2x)$$

$$f'_\alpha=\frac{1}{\sin^2x+\alpha^2\cos^2x}\cdot 2\alpha\cos^2x=\frac{2\alpha\cos^2x}{\sin^2x+\alpha^2\cos^2x}$$

$$I'_\alpha(\alpha)=\int_0^{\pi/2} f'_\alpha\,dx=\int_0^{\pi/2}\frac{2\alpha\cos^2x}{\sin^2x+\alpha^2\cos^2x}\,dx \quad \begin{matrix}/:\cos^2x\\ /:\cos^2x\end{matrix} \quad =2\alpha\int_0^{\pi/2}\frac{dx}{tg^2x+\alpha^2}$$

$$=\left|\begin{matrix} tg\,x=t & x=arctg\,t\\ x=0\Rightarrow t=0 & dx=\dfrac{dt}{1+t^2}\\ x=\frac{\pi}{2}\Rightarrow t=\infty & \end{matrix}\right|=2\alpha\int_0^{\infty}\frac{dt}{(t^2+\alpha^2)(t^2+1)}$$

$$\frac{1}{(x^2+\alpha^2)(x^2+1)}=\frac{Ax+B}{x^2+\alpha^2}+\frac{Cx+D}{x^2+1}\quad /(x^2+\alpha^2)(x^2+1)$$

$$1=A(x^3+x)+B(x^2+1)+C(x^3+\alpha^2x)+D(x^2+\alpha^2)$$

$A+\quad C=0 \quad (1)$

$B+\quad D=0 \quad (2)$

$A+\alpha^2C=0 \quad (3)$

$B+\alpha^2D=1 \quad (4)$

(1)-(4): $C-\alpha^2C=0 \Rightarrow C=0 \Rightarrow A=0$

(2)-(4): $D-\alpha^2D=-1 \qquad (\alpha^2-1)D=1$

$\alpha^2D-D=1 \qquad D=\dfrac{1}{\alpha^2-1} \Rightarrow B=\dfrac{-1}{\alpha^2-1}$

$$I'_\alpha(\alpha) = 2\alpha \int_0^\infty \frac{dx}{(x^2+\alpha^2)(x^2+1)} = \frac{-2\alpha}{\alpha^2-1}\int_0^\infty \frac{dx}{x^2+\alpha^2} + \frac{2\alpha}{\alpha^2-1}\int_0^\infty \frac{dx}{x^2+1} =$$

$$- \frac{2\alpha}{\alpha^2-1} \cdot \frac{1}{\alpha} \operatorname{arctg}\frac{x}{\alpha}\Big|_0^\infty + \frac{2\alpha}{\alpha^2-1} \operatorname{arctg} x\Big|_0^\infty =$$

$$= -\frac{2}{\alpha^2-1}\left(\frac{\pi}{2}-0\right) + \frac{2\alpha}{\alpha^2-1}\left(\frac{\pi}{2}-0\right) =$$

$$= -\frac{\pi}{\alpha^2-1} + \frac{\pi}{\cancel{2}}\cdot\frac{\cancel{2}\alpha}{\alpha^2-1} = \frac{\pi(\alpha-1)}{\underbrace{\alpha^2-1}_{(\alpha-1)(\alpha+1)}} = \frac{\pi}{\alpha+1}$$

$$I'_\alpha(\alpha) = \frac{\pi}{\alpha+1} \Rightarrow I(\alpha) = \pi \ln|\alpha+1| + C = \left|\text{kako je } \alpha>0\right| = \pi\ln(\alpha+1)+C \quad \ldots(*)$$

$$I(\alpha) = \int_0^{\pi/2} \ln(\sin^2 x + \alpha^2\cos^2 x)\,dx \Rightarrow I(1) = \int_0^{\pi/2} \ln(1)\,dx = 0 \quad \ldots(**)$$

$$I(1) \overset{(*)}{=} \pi\ln 2 + C \overset{(**)}{=} 0 \Rightarrow C = -\pi\ln 2$$

$$\int_0^{\pi/2} \ln(\sin^2 x + \alpha^2\cos^2 x)\,dx = \pi\ln(\alpha+1) - \pi\ln 2 = \pi\ln\frac{\alpha+1}{2}$$

traženo rješenje

Izračunati $I(\alpha)=\int_0^{\infty}\frac{1-e^{-\alpha x}}{xe^{x}}dx$ ako je $\alpha>-1$.

Rj.

$$I'(\alpha)=\int_a^b f'_\alpha(\alpha,x)\,dx$$

$$f(\alpha,x)=\frac{1-e^{-\alpha x}}{x\,e^{x}}\ ,\quad f'_\alpha=\frac{-e^{-\alpha x}\cdot(-x)}{xe^{x}}$$

$$I'(\alpha)=\int_0^{\infty}\frac{x\,e^{-\alpha x}}{xe^{x}}dx=\int_0^{\infty}e^{-\alpha x-x}dx=\int_0^{\infty}e^{-(\alpha+1)x}dx=\left|\begin{array}{l}-(\alpha+1)x=s\\ -(\alpha+1)dx=ds\\ dx=-\frac{ds}{\alpha+1}\\ x=0\Rightarrow s=0\\ x=\infty\Rightarrow s=-\infty\end{array}\right|=-\frac{1}{\alpha+1}\int_0^{-\infty}e^{s}ds=\frac{-1}{\alpha+1}e^{s}\Big|_0^{-\infty}=0-\frac{(-1)}{\alpha+1}=\frac{1}{\alpha+1}$$

$$I'_\alpha=\frac{1}{\alpha+1}\Rightarrow I(\alpha)=\int\frac{1}{\alpha+1}d\alpha=\ln|\alpha+1|+C$$

Kako je $I(0)=\int_0^{\infty}\frac{1-e^{0}}{xe^{x}}dx=0$ to je $I(0)=\ln 1+C=0$

$\Rightarrow C=0$

$I(\alpha)=\ln|\alpha+1|$ traženo rješenje

za vježbu Izračunati $I(\alpha)=\int_0^{\infty}\frac{1-e^{-\alpha x^2}}{xe^{x^2}}dx$, ako je $\alpha>-1$.

za vježbu Izračunati $\int_0^{\pi/2}\frac{\operatorname{arctg}(\alpha\operatorname{tg}x)}{\operatorname{tg}x}dx$

rješenje: $I(\alpha)=\frac{\pi}{2}\ln|1+\alpha|$.

Izračunati

$$\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}\,dx .$$

Rj.:

$$I'(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx, \qquad I(\alpha)=\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}\,dx$$

$$f(\alpha,x)=e^{-x}\frac{\sin\alpha x}{x}, \qquad f'_\alpha=\frac{e^{-x}}{x}\cdot x\cos\alpha x=e^{-x}\cos\alpha x$$

$$I'(\alpha)=\int_0^\infty e^{-x}\cos\alpha x\,dx=\lim_{R\to\infty}\int_0^R e^{-x}\cos\alpha x\,dx=$$

$$=\left|\begin{matrix} u=e^{-x} & dv=\cos\alpha x\,dx \\ du=-e^{-x}dx & v=\frac{1}{\alpha}\sin\alpha x\end{matrix}\right|=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-x}\sin\alpha x\Big|_0^R+\frac{1}{\alpha}\int_0^R e^{-x}\sin\alpha x\,dx\right)$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R+\frac{1}{\alpha}\int_0^R e^{-x}\sin\alpha x\,dx\right)=\left|\begin{matrix} u=e^{-x} & dv=\sin\alpha x\,dx \\ du=-e^{-x}dx & v=-\frac{1}{\alpha}\cos\alpha x\end{matrix}\right|=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R+\frac{1}{\alpha}\left(-\frac{1}{\alpha}\underbrace{e^{-x}\cos\alpha x\Big|_0^R}_{(e^{-R}\cos\alpha R-e^{0}\cdot 1)}-\frac{1}{\alpha}\int_0^R e^{-x}\cos\alpha x\,dx\right)\right)=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\sin\alpha R-\frac{1}{\alpha^2}e^{-R}\cos\alpha R+\frac{1}{\alpha^2}-\frac{1}{\alpha^2}\int_0^R e^{-x}\cos\alpha x\,dx\right)=$$

$$=\lim_{R\to\infty}\left(\frac{1}{\alpha}e^{-R}\underbrace{\sin\alpha R}_{\text{ovo je između -1 i 1}}-\frac{1}{\alpha^2}e^{-R}\underbrace{\cos\alpha R}_{\text{uzima vrijednosti između -1 i 1}}+\frac{1}{\alpha^2}\right)-\frac{1}{\alpha^2}\underbrace{\lim_{R\to\infty}\int_0^R e^{-x}\cos\alpha x\,dx}_{I'(\alpha)}$$

Sad imamo

$$\left(1+\frac{1}{\alpha^2}\right)\int_0^\infty e^{-x}\cos\alpha x\,dx=\frac{1}{\alpha^2} \Rightarrow \int_0^\infty e^{-x}\cos\alpha x\,dx=\frac{\frac{1}{\cancel{\alpha^2}}}{\frac{\alpha^2+1}{\cancel{\alpha^2}}}=\frac{1}{\alpha^2+1}$$

Kako je $I'(\alpha)=\frac{1}{\alpha^2+1}$ to je $I(\alpha)=\int\frac{1}{\alpha^2+1}\,d\alpha=\operatorname{arctg}\alpha+C$

$I(0)=0=\operatorname{arctg}0+C \Rightarrow C=0$

Prema tome $\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}\,dx=\operatorname{arctg}\alpha$ traženo rješenje

Zadaci za vježbu

3730. Naći oblast definisanosti funkcije $f(x)=\int\limits_0^1 \frac{dz}{\sqrt{x^2+z^2}}$.

3731. Naći krivinu krive $y=\int\limits_{\pi}^{2\pi} \frac{\sin\alpha x}{\alpha}\,d\alpha$ u tački čija je apscisa $x=1$.

3732. Polazeći od jednakosti $\int\limits_0^b \frac{dx}{1+ax}=\frac{1}{a}\ln(1+ab)$ izvesti diferenciranjem po parametru, sledeću formulu:

$$\int\limits_0^b \frac{x\,dx}{(1+ax)^2}=\frac{1}{a^2}\ln(1+ab)-\frac{b}{a(1+ab)}.$$

3733. Polazeći od jednakosti $\int\limits_0^b \frac{dx}{x^2+a^2}=\frac{1}{a}\operatorname{arctg}\frac{b}{a}$, izračunati integral $\int\limits_0^b \frac{dx}{(x^2+y^2)^3}$.

3734. Polazeći od jednakosti $\int\limits_0^\infty \frac{dx}{a^2+x^2}=\frac{\pi}{2a}$. izračunati $\int\limits_0^\infty \frac{dx}{(x^2+a^2)^n}$ (n je ceo pozitivan broj).

3735. Izračunati vrednost integrala $\int\limits_0^\infty e^{-ax}x^{n-1}\,dx$ (n je ceo pozitivan broj) za $a>0$, našavši prethodno vrednost $\int\limits_0^\infty e^{-ax}\,dx$.

3736*. Polazeći od jednakosti (vidi zad. 2318)

$$\int\limits_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi}{2|ab|},\quad \text{naći} \int\limits_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}.$$

Rješenja

3730. Definisana je za sve vrednosti $x\neq 0$. **3731.** 3π.

3733. $\frac{b}{8a^4}\left\{\frac{5a^2+3b^2}{(a^2+b^2)^2}+\frac{3}{ab}\operatorname{arctg}\frac{b}{a}\right\}$. **3734.** $\frac{1\cdot 3\cdot 5\ldots(2n-3)}{2\cdot 4\cdot 6\ldots(2n-2)}\,\frac{\pi}{2a^{2n-1}}$ $(n>1)$.

3735. $\frac{(n-1)!}{a^n}$. **3736*.** $\frac{\pi(a^2+b^2)}{4|ab|^3}$. Diferencirati po a ili b i sabrati rezultate.

U zadacima 3737 — 3749 izračunati vrednosti datih integrala metodom diferenciranja po parametru.

3737. $\int_0^{\infty} \frac{1-e^{-ax}}{xe^x} dx \quad (a>-1)$.

3738. $\int_0^{\infty} \frac{1-e^{-ax^2}}{xe^{x^2}} dx \quad (a>-1)$.

3739. $\int_0^{1} \frac{\operatorname{arctg} ax}{x\sqrt{1-x^2}} dx$.

3740. $\int_0^{1} \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}} dx \quad (a^2<1)$.

3741. $\int_0^{\infty} \frac{\operatorname{arctg} ax}{x(1+x^2)} dx$.

3742. $\int_0^{1} \frac{\ln(1-a^2x^2)}{\sqrt{1-x^2}} dx \quad (a^2<1)$.

3743. $\int_0^{\pi} \frac{\ln(1+a\cos x)}{\cos x} dx \quad (a^2<1)$.

3744. $\int_0^{\frac{\pi}{2}} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x} \quad (a^2<1)$.

3745. $\int_0^{\infty} \frac{1-e^{-ax^2}}{x^2} dx \quad (a>0)$, znajući da je

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a>0) \text{ (vidi zadatak 2439)}.$$

3746*. $\int_0^{\infty} \frac{e^{-ax^2}-e^{-bx^2}}{x^2} dx \quad (a>0,\ b>0)$.

3747*. $\int_0^{\infty} e^{-ax}\frac{\sin bx-\sin cx}{x} dx \quad (a>0)$.

3748. $\int_0^{\infty} e^{-ax}\frac{\cos bx-\cos cx}{x} dx \quad (a>0)$.

3749*. $\int_0^{\frac{\pi}{2}} \ln(a^2\cos^2 x+b^2\sin^2 x)\, dx$.

Rješenja

3737. $\ln(1+a)$. **3738.** $\frac{1}{2}\ln(1+a)$.

3739. $\frac{\pi}{2}\ln(a+\sqrt{1+a^2})$.

3740. $\pi(\sqrt{1-a^2}-1)$.

3741. $\frac{\pi}{2}\ln(1+a)$, ako je $a>0$; $-\frac{\pi}{2}\ln(1-a)$, ako je $a<0$.

3742. $\pi\ln\frac{1+\sqrt{1-a^2}}{2}$.

3743. $\pi \arcsin a$. **3744.** $\pi\arcsin a$.

3745. $\sqrt{\pi a}$.

3746*. $\sqrt{\pi}(\sqrt{b}-\sqrt{a})$.

Naći izvode po a ili po b.

3747*. $\operatorname{arctg}\frac{b}{a}-\operatorname{arctg}\frac{c}{a}=\operatorname{arctg}\frac{a(b-c)}{a^2+bc}$.

Diferencirati po b ili po c.

3748. $\frac{1}{2}\ln\frac{a^2+b^2}{a^2+c^2}$.

3749*. $\pi\ln\frac{a+b}{2}$. Diferencirati po a ili po b.

Zadaci za vježbu

3750. Izračunavši integral $\int\limits_0^{\frac{\pi}{2}} \frac{\operatorname{arctg}(a \operatorname{tg} x)}{\operatorname{tg} x}\, dx$, naći $\int\limits_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} x}\, dx$.

3751. Koristeći jednakost $\int\limits_0^1 x^n\, dx = \frac{1}{n+1}$, izračunati integral

$$\int\limits_0^1 \frac{x^\beta - x^\alpha}{\ln x}\, dx \quad (\alpha > -1,\ \beta > -1).$$

3752. Koristeći jednakost $2a\int\limits_0^\infty e^{-a^2x^2}\, dx = \sqrt{\pi}$ (vidi zadatak 2439), izračunati integral $\int\limits_0^\infty \left(e^{-\frac{a^2}{x^2}} - e^{-\frac{b^2}{x^2}}\right) dx$.

3753. Iz relacije $\int\limits_0^\infty e^{-z^2}\, dz = \frac{\sqrt{\pi}}{2}$ (Puasonov integral) izvesti jednakost

$$\frac{1}{\sqrt{x}} = \frac{2}{\sqrt{\pi}} \int\limits_0^\infty e^{-z^2 x}\, dz\ (x > 0$$

i iskoristiti je za izračunavanje integrala (integral difrakcije ili Frenelov integral):

$$\text{a)}\ \int\limits_0^\infty \frac{\cos x\, dx}{\sqrt{x}}; \qquad \text{b)}\ \int\limits_0^\infty \frac{\sin x\, dx}{\sqrt{x}}.$$

Rješenja

3750. $\frac{\pi}{2}\ln(1+a)$, ako je $a>0$; $-\frac{\pi}{2}\ln(1-a)$, ako je $a<0$; $\int\limits_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} x}\, dx = \frac{\pi}{2}\ln 2$.

3751*. $\ln\frac{1+\beta}{1+\alpha}$. Integrirati po parametru n u granicama od α do β.

3752. $\sqrt{\pi}\,(b-a)$. **3753.** $\int\limits_0^\infty \frac{\cos x\, dx}{\sqrt{x}} = \int\limits_0^\infty \frac{\sin x\, dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}$.

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
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